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Translated by J. J. D.

UDC 532.516

STABILITY AND BIFURCATION OF COUETTE FLOW IN THE CASE OF A NARROW GAP BETWEEN ROTATING CYLINDERS

PMM Vol. 38, №6, 1974, pp.1025-1030

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(Received March 18, 1974)

Stability and bifurcation of Couette flow between concentric rotating cylinders are investigated for the case when the ratios of their radii R and angular velocities Ω are nearly equal to unity. The limiting problem in the linear theory when $R \to 1$ and $\Omega \to 1$ is the problem of convection stability in the layer [1]. We find that this is also correct in the case of a nonlinear problem. Below we show that solution of the problem of free convection yields the principal term of the expansion of the secondary flow (Taylor vortex) in the powers of a small parameter $\delta = R - 1$. Therefore the results of [2, 3] can be used to provide, in the present case, a strict justification for the use of the Liapunov-Schmidt method to compute the Taylor vortices. The numerical results obtained for the critical Reynolds' number and the amplitude of the secondary flow provide a good illustration of the asymptotic passage as $\delta \to 0$.

1. Statement of the problem. Let a viscous incompressible fluid of unit density fill the space between two infinite concentric cylinders of radii R_1 and R_2, rotating at the angular velocities Ω_1 and Ω_2. Let $R \to 1$ and $\Omega \to 1$, so that

$$(\Omega - 1) / (R - 1) = c = \text{const}, \quad R = R_2 / R_1, \ \Omega = \Omega_2 / \Omega_1$$

We choose $R_2 - R_1$ as the characteristic length and $\Omega_1 (R_2 - R_1)$ as the characteristic angular velocity. Then the Couette flow assumes, in the cylindrical coordinates (r, θ, z), the following form:

$$\mathbf{v}_0 = (0, v_{0\theta}, 0), \quad v_{0\theta} = ar + b / r \tag{1.1}$$
$$a = \frac{\Omega R^2 - 1}{R^2 - 1}, \quad b = -\frac{(\Omega - 1) R^2}{(R^2 - 1)\delta^2}, \quad \mathrm{Re} = \frac{\Omega_1 (R_2 - R_1)^2}{\nu} = \frac{\delta^2 \Omega_1^2 R_1^2}{\nu}$$

Here $\delta = R - 1$ is a small parameter when $R \to 1$. A strict proof exists [4, 5] of the fact that the Couette flow is unstable when the Reynolds' number Re is sufficiently large. The value Re_* at which loss of stability takes place is called the critical value.

We shall seek a steady state, rotationally symmetric solution of the Navier-Stokes equations, different from (1.1), in the form

$$\mathbf{v}_1 = \mathbf{v}_0 + \mathbf{v}, \quad p_1 = p_0 + p \tag{1.2}$$

(where p_0 is the pressure corresponding to the Couette flow), for the supercritical values of $\mathrm{Re} = \mathrm{Re}_* + \varepsilon^2$. Substituting (1.2) into the Navier-Stokes equations, we obtain the following relations for $\mathbf{v}$ and p :

$$\nabla^2 v_r + \frac{\delta}{1 + \rho\delta}\frac{\partial v_r}{\partial \rho} - \frac{\delta^2}{(1 + \rho\delta)^2} v_r - \frac{\partial p}{\partial \rho} = (\mathrm{Re}_* + \varepsilon^2) \times \tag{1.3}$$
$$\left[(\mathbf{v}, \nabla) v_r - 2\frac{\delta v_{0\theta}}{1 + \rho\delta} v_\theta - \frac{\delta}{1 + \rho\delta} v_\theta^2\right]$$
$$\nabla^2 v_\theta + \frac{\delta}{1 + \rho\delta}\frac{\partial v_\theta}{\partial \rho} - \frac{\delta^2}{(1 + \rho\delta)^2} v_\theta = (\mathrm{Re}_* + \varepsilon^2) \times$$
$$\left[(\mathbf{v}, \nabla) v_\theta + 2a v_r + \frac{\delta}{1 + \rho\delta} v_r v_\theta\right]$$
$$\nabla^2 v_z - \frac{\partial p}{\partial z} = (\mathrm{Re}_* + \varepsilon^2)(\mathbf{v}, \nabla) v_z$$
$$\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{\delta}{1 + \delta\rho} v_r = 0$$
$$\mathbf{v} = 0 \quad \text{for } \rho = 0, 1$$
$$\rho = \frac{(r\delta - 1)}{\delta}, \quad \nabla = \left(\frac{\partial}{\partial \rho}, 0, \frac{\partial}{\partial z}\right), \quad \mathbf{v} = (v_r, v_\theta, v_z)$$

We expand the function $v_{0\theta}(r)$ into the following series in small parameter δ

$$v_{0\theta} = \frac{1}{\delta} + (c + 1)\rho + \sum_{n=1}^{\infty} v_n \delta^n, \quad a = \sum_{n=0}^{\infty} a_n \delta^n, \quad b = \sum_{n=0}^{\infty} b_n \delta^{n-2}$$
$$v_n = a_{n+1} + \rho a_n + \sum_{m=0}^{n+1} b_{n-m+1}(-\rho)^m, \quad n = 1, 2, \ldots$$
$$a_0 = (c + 2) / 2, \quad b_0 = -c / 2, \quad a_1 = -b_1 = 3c / 4$$
$$a_n = -b_n = (-1)^n c / 2^{n+1}, \quad n = 2, 3, \ldots$$

2. The asymptotics of the problem on stability with $\delta \to 0$. We shall show that the solution of the system (1.3) is analytic in ε and δ in the neighborhood of $\varepsilon = 0$ and $\delta = 0$. Let us investigate the resulting situation using a more general approach.

Let the following operator equation be given in the Hilbert space H :

$$A_0\mathbf{u} + A_1\mathbf{u} - \lambda K\mathbf{u} = \lambda L(\mathbf{u}, \mathbf{u}) \tag{2.1}$$

Here A_0 is a self-conjugate, positive-definite linear operator, A_1 and K are linear operators and L is a bilinear operator; A_0, A_1, K and L are, generally speaking, unbounded. Let $A_0^{-1}A_1$ and $A_0^{-1}K$ be completely continuous on H_1 and $A_0^{-1}L(\mathbf{u}, \mathbf{v})$ completely continuous on $H_1 \oplus H_1$. Here H_1 denotes the energy space of the operator A_0, i.e. it is the closure of the domain of definition of A_0 in the metric $(\mathbf{u}, \mathbf{v})_{H_1} = (A_0\mathbf{u}, \mathbf{v})_H$. Let λ_* be a simple eigenvalue, φ the corresponding eigenvector of the linear problem

$$A_0\varphi + A_1\varphi - \lambda_* K\varphi = 0 \tag{2.2}$$

and ψ the eigenvector of the conjugate problem. Since λ_* is simple, we can assume that $(\varphi, \psi)_H = 1$. Setting $\lambda = \lambda_* + \varepsilon^2$ and inverting the operator A_0, we reduce (2.1) to the form

$$\mathbf{u} + A_0^{-1}A_1\mathbf{u} - \lambda_* A_0^{-1}K\mathbf{u} = \varepsilon^2 A_0^{-1}K\mathbf{u} + (\lambda_* + \varepsilon^2)A_0^{-1}L(\mathbf{u}, \mathbf{u}) \tag{2.3}$$

where all operators are completely continuous.

Lemma. Let the following conditions hold:

1) operators A_1, K and L are analytic functions of the small parameter δ in the region $|\delta| < \delta_0$ and A_0 is independent of δ.

2) for any value of δ the eigenvalue $\lambda_*(\delta)$ is simple and

$$(L(\varphi, \varphi), \psi)_H = 0$$

3) $C_{20} > 0$ when $\delta = 0$, where C_{20} is a constant defined by (2.10). Then for small values of δ, exactly two small solutions of (2.3) exist (which tend to zero when $\varepsilon \to 0$). Both solutions are analytic in ε and δ in the neighborhood of the point (0, 0).

Proof. We shall seek a solution of (2.3) in the form

$$\mathbf{u} = \gamma\varphi + \mathbf{v}, \qquad (\mathbf{v}, \psi)_H = 0$$

From (2.3) we obtain

$$(I + A_0^{-1}A_1 - \lambda_* A_0^{-1}K)\,\mathbf{v} = \varepsilon^2 A_0^{-1}K\mathbf{v} + \varepsilon^2\gamma A_0^{-1}K\varphi + (\lambda_* + \varepsilon^2)A_0^{-1}L(\mathbf{v} + \gamma\varphi, \mathbf{v} + \gamma\varphi) \equiv \mathbf{f} \tag{2.4}$$

Let us introduce the space H_0 of vectors $\mathbf{u} \in H$ for which $(\mathbf{u}, \psi)_H = 0$. We note that the operator $(I + A_0^{-1}A_1 - \lambda_* A_0^{-1}K)$ acts from H into H_0. For any $\mathbf{f} \in H_0$ Eq. (2.4) has a solution, i.e. an operator N exists such that

$$\mathbf{v} = N\mathbf{f}, \qquad (\mathbf{f}, \psi)_H = 0 \tag{2.5}$$

The operator N (see [6], chap. 7, Sect. 6) is analytic in δ. Substituting the expression for $\mathbf{f}$ into (2.5), we obtain a system of equations for $\mathbf{v}$ and γ which is equivalent to (2.4)

$$\mathbf{v} = N(\varepsilon^2 A_0^{-1}K\mathbf{v} + \varepsilon^2\gamma A_0^{-1}K\varphi + (\lambda_* + \varepsilon^2)A_0^{-1}L(\mathbf{v} + \gamma\varphi, \mathbf{v} + \gamma\varphi)) \equiv T\mathbf{v} \tag{2.6}$$

$$(\varepsilon^2 A_0^{-1}K\mathbf{v} + \varepsilon^2\gamma A_0^{-1}K\varphi + (\lambda_* + \varepsilon^2)A_0^{-1}L(\mathbf{v} + \gamma\varphi, \mathbf{v} + \gamma\varphi), \psi)_H = 0 \tag{2.7}$$

When ε and γ are sufficiently small, the operator T is a contraction operator in any sphere in H_1 of sufficiently small radius and with its origin at zero. In accordance with the theorem on implicit functions [7] the solution $\mathbf{v}$ of (2.6) can be sought in the form

$$\mathbf{v} = \sum_{k,l=0}^{\infty} \mathbf{v}_{kl}\varepsilon^k\gamma^l, \qquad \mathbf{v}_{00} = 0 \tag{2.8}$$

The coefficients $\mathbf{v}_{kl}$ can be found from (2.6) and (2.7), e.g.

$$\mathbf{v}_{k0} = 0, \ \mathbf{v}_{01} = 0, \ \mathbf{v}_{02} = \lambda_* N A_0^{-1} L(\varphi, \varphi), \ \mathbf{v}_{11} = 0, \ \ldots$$

Substituting (2.8) into (2.7) we obtain

$$\gamma^2 + \zeta^2 = 0 \qquad (2.9)$$

$$\zeta = (C_{20}\,\varepsilon^2 + C_{03}\,\gamma^3 + C_{12}\gamma\varepsilon^2 + O(\gamma^i \varepsilon^j))^{1/2}, \ i + j = 4$$

The constants C_{ij} can be expressed in terms of φ, ψ and $\mathbf{v}_{kl}$, e.g.

$$C_{20} = \frac{(A_0^{-1}K\varphi, \psi)_H}{\lambda_* (A_0^{-1}L(\mathbf{v}_{02}, \varphi) + A_0^{-1}L(\varphi, \mathbf{v}_{02}), \psi)_H} \qquad (2.10)$$

Solution of (2.9) reduces to solving the following equations:

$$\Phi_1(\mathbf{v}, \gamma, \varepsilon, \delta) \equiv \gamma + \zeta = 0 \qquad (2.11)$$

$$\Phi_2(\mathbf{v}, \gamma, \varepsilon, \delta) \equiv \gamma - \zeta = 0 \qquad (2.12)$$

The system of equations (2.6) and (2.11) for determining $\mathbf{v}$ and γ can be written in the form

$$\mathbf{F}(\mathbf{z}, \varepsilon, \delta) = 0 \qquad (2.13)$$

$$\mathbf{z} = (\mathbf{v}, \gamma) \in H_1 \oplus R, \qquad \mathbf{F} = (I - T, \ \Phi_1)$$

When $\delta = 0$, the condition $C_{20} > 0$ implies that a solution $\mathbf{z}(\varepsilon, 0) = (\mathbf{v}(\varepsilon), \gamma(\varepsilon))$ of (2.13) exists which is analytic in ε, and $\mathbf{z}(0, 0) = 0$. With $\varepsilon = 0$ and arbitrary δ, we find that $\mathbf{z}(0, \delta) = 0$ is a solution of (2.13). The eigenvalue $\lambda_*(\delta)$ and the corresponding eigenvectors φ and ψ are analytic in δ by virtue of the fact that $\lambda_*(\delta)$ is simple [6]. Thus $\mathbf{F}$ is analytic in ε and δ in the neighborhood of $\mathbf{z} = 0, \varepsilon = 0, \delta = 0$, $\mathbf{F}(0, 0, 0) = 0$ and

$$\mathbf{F}_\mathbf{z}'(0, 0, 0) = \begin{pmatrix} (\mathbf{v} - T\mathbf{v})_\mathbf{v}' & (\mathbf{v} - T\mathbf{v})_\gamma' \\ \Phi_{1\mathbf{v}}' & \Phi_{1\gamma}' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using the theorem on implicit functions [7] we find that a unique solution of (2.13) exists, analytic in ε and δ in the neighborhood of $\varepsilon = 0$, $\delta = 0$. The second solution is found in the similar manner from the system (2.6), (2.12), and this completes the proof of the lemma.

The system (1.3) can be written in the form (2.1) by setting

$$A_0 = \Pi\nabla^2, \quad A_1 = \Pi\left(\frac{\delta}{1 + \rho\delta} G_1\right), \quad K = 2\Pi G_2$$

$$L(\mathbf{v}, \ \mathbf{v}) = \Pi\left((\mathbf{v}, \nabla)\mathbf{v} + \frac{\delta}{1 + \rho\delta} G_3\mathbf{v}\right)$$

$$G_1\mathbf{v} = \left(\frac{\partial v_r}{\partial \rho} - \frac{\delta}{1 + \rho\delta} v_r, \ \frac{\partial v_\theta}{\partial \rho} - \frac{\delta}{1 + \rho\delta} v_\theta, \ 0\right)$$

$$G_2\mathbf{v} = \left(-\frac{\delta v_{0\theta}}{1 + \rho\delta} v_\theta, \ a v_r, 0\right), \quad G_3\mathbf{v} = (-v_\theta^2, \ v_r v_\theta, 0)$$

Here Π is an operator of orthogonal projection of the solenoidal vectors with a null normal components at the boundary, on a subspace of $W_2^{(2)}$. The domain of definition of the operstors A_0, A_1, L and K is a set of solenoidal vectors belonging to $W_2^{(2)}$, vanishing at the boundary.

The lemma implies that the solution of (1.3) is analytic in δ and ε in the neighborhood of $\varepsilon = 0$, $\delta = 0$. This solution can conveniently be sought in the form

$$\mathbf{v} = \sum_{n=0}^{\infty} \mathbf{u}_n \delta^n, \quad p = \sum_{n=0}^{\infty} p_n \delta^n \tag{2.14}$$

$$\mathbf{u}_n = \sum_{k=0}^{\infty} \mathbf{u}_{nk} \varepsilon^k, \quad p_n = \sum_{k=0}^{\infty} p_{nk} \varepsilon^k \tag{2.15}$$

The Reynolds' number $\mathrm{Re} = \mathrm{Re}_* + \varepsilon^2$ can also be written in the form of the series

$$\mathrm{Re} = \sum_{n=0}^{\infty} \mathrm{Re}_{*n} \delta^n + \varepsilon^2 \tag{2.16}$$

We introduce the stream function ψ_0 and the function τ_0 as follows:

$$u_{0r} = \frac{1}{\mathrm{Re}_0} \frac{\partial \psi_0}{\partial z}, \quad u_{0z} = -\frac{1}{\mathrm{Re}_0} \frac{\partial \psi_0}{\partial \rho}, \quad u_{0\theta} = 2a_0 \tau_0$$

Then from (1.3), (2.14) and (2.16) we obtain the expressions for ψ_0 and τ_0

$$\nabla^4 \psi_0 = -4(\mathrm{Re}_0)^2 a_0 \frac{\partial \tau_0}{\partial z} + [\psi_0, \nabla^2 \psi_0] \tag{2.17}$$

$$\nabla^2 \tau_0 = \frac{\partial \psi_0}{\partial z} + [\psi_0, \tau_0]$$

$$\frac{\partial \psi_0}{\partial z} = \frac{\partial \psi_0}{\partial \rho} = \tau_0 = 0 \quad \text{for} \quad \rho = 0,1$$

$$[f, \varphi] = \frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial \rho} - \frac{\partial f}{\partial \rho} \frac{\partial \varphi}{\partial z}$$

The system (2.17) coincides with the nonlinear perturbation problem in the case of a convective motion in a liquid layer heated from below if $-4(\mathrm{Re}_0)^2 a_0$ is taken as the Rayleigh number and the Prandtl number is assumed equal to unity. Thus the solution of (1.3) coincides with the accuracy of up to the infinitesimals of the order of δ, with the solution of the nonlinear problem of convection stability in a layer.. The problems of investigating the system (2.17) and computing the convective motions in a fluid were studied by a number of authors (see [8], chap. 1, Sect. 29).

By virtue of the analyticity, we obtain

$$C_{20} = g_* + \delta g_{*1} + \delta^2 g_{*2} + \ldots \tag{2.18}$$

The constant g_* corresponds to the problem of convection and a rigorous proof that $g_* > 0$ is given in [2]. From (2.18) it follows that $C_{20} > 0$ for sufficiently small δ. Using the results of [2, 3], we arrive at the following theorem.

Theorem. If the gap between the rotating cylinders is sufficiently small and their angular velocities sufficiently close to each other, then the Couette flow becomes unstable when the Reynolds number passes through its critical value. A new, steady state flow appears, unique to within the translation along the cylinders' axis, stable at $\mathrm{Re} > \mathrm{Re}_*$ and analytic in the small parameter $\varepsilon = (\mathrm{Re} - \mathrm{Re}_*)^{1/2}$.

3. Numerical computations. A secondary solution $2\pi / \alpha$-periodic in z was sought for a finite value of δ, in the form of the Liapunov-Schmidt series

$$\mathbf{v} = \sum_{k=1}^{\infty} \varepsilon^k \mathbf{v}_k, \qquad \mathbf{v}_1 = \beta_1 \varphi$$

where φ is the eigenvector of the linearized problem corresponding to (1.3). The vector φ satisfies the normalization condition

$$\int_{-\pi/2\alpha}^{\pi/2\alpha}\int_{1}^{R} \varphi_r(r, z)\, r\, dr\, dz = \frac{2}{\alpha}$$

The amplitude β_1 and Re_* were computed using the method given in [3]. The asymptotic values of β_1 and Re_* for $\delta \to 0$ were found using the results of the computations for the convective motion of a fluid in a layer. All computations were performed for $\alpha = 3.115$ and $\Omega = 1 / R^2 - 0.05\delta$, and the results are

$R = 1$	1.15	1.2	1.25	1.3	1.35	1.5
$\mathrm{Re}_* = 130.68$	126.95	126.05	125.29	124.66	124.14	123.18
$\beta_1 \cdot 10^3 = 0.0217$	0.0226	0.0555	0.1118	0.1990	0.3248	1.0175

When $\delta \to 0$, the values of β_1 and Re_* tend to their limiting values.

The authors thank I. V. Bulaev and G. K. Ter-Grigor'iants for permission to use their results.

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Translated by L. K.